V. Functions of Two or More Variables

Limits:

An motivating example: $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = ?$

i.e. $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}} \longrightarrow ?$ as $(x,y) \longrightarrow (0,0)$

Observations:

1° Though $(x,y)\to(0,0)$, we couldn't just plug in $x=0$ & $y=0$ to get the limit as $f(0,0) = \dfrac{(0)(0)}{\sqrt{0^2+0^2}} = \dfrac{0}{0}$ which is not even defined (or indeterminate)

$\therefore$ It is not obvious where $f(x,y) \longrightarrow ?$ as $(x,y)\to(0,0)$.

2° However, if we observe the difference between $f(x,y)$ and 0, we have

$$|f(x,y)-0| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \frac{|x||y|}{\sqrt{x^2+y^2}} \leq \frac{1}{2}\sqrt{x^2+y^2} = \frac{1}{2}|(x,y)-(0,0)|$$

(note that all these computations are justified as long as $(x,y)\neq(0,0)$)

Thus, $\forall (x,y)\neq(0,0)$, $|f(x,y)-0| \leq \frac{1}{2}|(x,y)-(0,0)|$

i.e. we could control the difference or distance between $f(x,y)$ and 0 by the distance of (x,y) from $(0,0)$. $\Rightarrow$ as $(x,y)\to(0,0)$ but $(x,y)\neq(0,0)$, $f(x,y)\to 0$. We say that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$,

the limit of $f(x,y)$ is zero as $(x,y)\to(0,0)$ (but not equal to $(0,0)$)

Remark: (i) When we talk about the limit of $f(x,y)$ as $(x,y)\to(a,b)$, we are refering to the asymptotic behavior of $f(x,y)$ or where $f(x,y)$ tends to? as (x,y) approaches the pt. (a,b) but not equal to (a,b).

(ii) the limit of $f(x,y)$ as $(x,y)\to(a,b)$ has nothing to do with the behavior of $f(x,y)$ at $(x,y)\equiv(a,b)$, as motivated by the above example, $f(a,b)$ may not even be defined.

(iii) The limit of $f(x,y)$ should be independent of the way that (x,y) approaches (a,b).

Def'n. $\lim_{(x,y)\to(a,b)} f(x,y) = l$ iff $\forall \varepsilon > 0$ however small, there exists $\delta > 0$, depending on ε such that whenever $0 < |(x,y)-(a,b)| = \sqrt{(x-a)^2+(y-b)^2} < \delta$, $|f(x,y) - l| < \varepsilon$.

In non-technical terms, $\lim_{(x,y)\to(a,b)} f(x,y) = l$ iff we could make $f(x,y)$ as close as we want to l by making (x,y) sufficiently close to (a,b) but not equal to (a,b).

Ex. Prove $\lim_{(x,y)\to(2,4)} xy = (2)(4) = 8$ base on ε-δ definition of limit.

Pf:

$$|f(x,y)-8| = |xy-8|$$

But
$$\begin{aligned} xy-8 &= (x-2+2)(y-4+4) - 8 \\ &= (x-2)(y-4) + 4(x-2) + 2(y-4) + \cancel{8} - \cancel{8} \end{aligned}$$

$$\begin{aligned} \therefore |f(x,y)-8| &= |(x-2)(y-4) + 4(x-2) + 2(y-4)| \\ &\le |x-2||y-4| + 4|x-2| + 2|y-4| \\ &\le |(x,y)-(2,4)|^2 + 6|(x,y)-(2,4)|. \end{aligned}$$

For $0 < |(x,y)-(2,4)| < \delta$

$|f(x,y)-8| \le \delta^2 + 6\delta \le 7\delta$ (where wlog, we could assume $0<\delta<1$)

Therefore, for any $\varepsilon > 0$, if we pick $\delta = \varepsilon/7$, whenever $0 < |(x,y)-(2,4)| < \delta$, $|f(x,y)-8| \le 7\delta = \varepsilon$. Hence $\lim_{(x,y)\to(2,4)} xy = 8$.

Remarks:

(i) In the proof, the key point is to control $|f(x,y) - l|$ by $|(x,y)-(a,b)|$

(ii) Unlike the 1st example where $f(x,y)$ is not even defined at the point (a,b), we have $\lim_{(x,y)\to(2,4)} f(x,y) = f(2,4)$ i.e. the limit of $f(x,y)$ as $(x,y)\to(a,b)$ is the same as $f(a,b)$, the value of f at (a,b).

Non ε-δ approach:

Back to the case $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

We could have use polar co-ordinate $x = r\cos\theta$, $y = r\sin\theta$

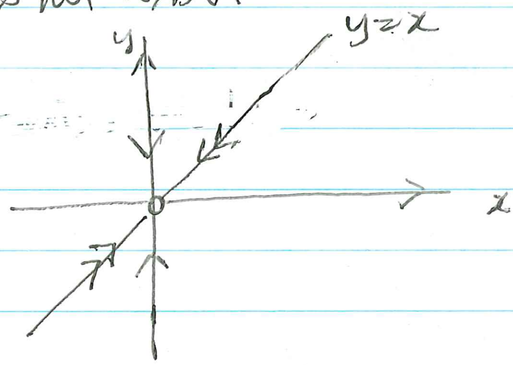
Then $|f(x,y) - 0| = \left|\frac{r^2\cos\theta\sin\theta}{r}\right| \le r$

as $(x,y) \to (0,0)$ i.e. $r \to 0$, $|f(x,y) - 0| = r \to 0$ //.

Non-existence of Limit

We remark once that $\lim_{(x,y)\to(a,b)} f(x,y)$ if exists, should be independent of the path that (x,y) approaches the pt. (a,b). Therefore, one way to show $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist is to show that $f(x,y)$ approaches different values as $(x,y) \to (a,b)$ along different paths.

Ex. Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.



Indeed, $\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x=0}} \frac{xy}{x^2+y^2} = \frac{0}{y^2} = 0$

But $\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=x}} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{y^2}{y^2+y^2} = 1/2$.

Donot match, $\therefore \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist //.

Definition (Continuity) $f(x,y)$ is said to be continuous at the point (a,b) iff $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Remark: This amounts to three different conditions.
(i) $f(a,b)$ is defined (ii) $\lim_{(x,y)\to(a,b)} f(x,y)$ exists (iii) They are equal.

Ex $f(x,y) = xy$, $\lim_{(x,y)\to(2,4)} f(x,y) = \lim_{(x,y)\to(2,4)} xy = 8 = f(2,4)$. $\therefore f$ is continuous at $(2,4)$.

Ex $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{for } (x,y) \neq (0,0) \\ 1 & \text{at } (0,0). \end{cases}$

In this case, $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0$

$f(0,0) = 1$

They don't match $\therefore f$ is discontinuous at $(0,0)$.

Ex. $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 1 & \text{at } (0,0) \end{cases}$

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist $\Rightarrow f$ is discontinuous at $(0,0)$.

Remarks: (i) However, in example 2, we could simply redefine $f(0,0) = 0$ (instead of 1), then f becomes continuous at $(0,0)$.
In general, if $\lim_{(x,y)\to(a,b)} f(x,y)$ exists, then we could always redefine $f(a,b)$ to agree with the limit. In this case, $f(x,y)$ is said to be having a Removable Discontinuity at (a,b).

(ii) As for the 3rd example, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, consequently we could not redefine $f(a,b)$ to match the limit. In this case f is said to have a non-removable discontinuity at (a,b).

Thm. (Basic Laws on Limits). Assuming $\lim_{(x,y)\to(a,b)} f(x,y) = l_1$ and $\lim_{(x,y)\to(a,b)} g(x,y) = l_2$,

then we have,

(i) $\lim_{(x,y)\to(a,b)} f(x,y) \pm g(x,y) = \lim_{(x,y)\to(a,b)} f(x,y) \pm \lim_{(x,y)\to(a,b)} g(x,y) = l_1 \pm l_2$.

(ii) $\lim_{(x,y)\to(a,b)} cf(x,y) = c\lim_{(x,y)\to(a,b)} f(x,y) = cl_1$.

(iii) $\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = \left(\lim_{(x,y)\to(a,b)} f(x,y)\right)\left(\lim_{(x,y)\to(a,b)} g(x,y)\right) = l_1 l_2$.

(iv) $\lim_{(x,y)\to(a,b)} f(x,y)/g(x,y) = \dfrac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)} = \dfrac{l_1}{l_2}$ provided $\lim_{(x,y)\to(a,b)} g(x,y) = l_2 \neq 0$.

Thm. (Elementary Continuous functions)

All the polynomials, sine, cosine and exponential functions are continuous in their own arguments. Their composite functions are continuous as well.

Ex. $\lim_{(x,y)\to(2,2)} \dfrac{\cos(x^2+y^2)}{1-x^2-y^2} = \dfrac{\cos(8)}{1-4-4} = \dfrac{-\cos 8}{7}$ //.

Ex $\lim_{(x,y,z)\to(1,1,0)} \dfrac{xy-z}{\cos(xyz)} = \dfrac{(1)(1)-0}{\cos 0} = 1$ //.

Partial Derivatives

Definition: Given $z=f(x,y)$, we define

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}(x,y) = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}(x,y) = f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

respectively to be the partial derivative of f with respect to x and partial deriv. of f wrt. y.

Essence: When we differentiate f partially with respective to x, it would be like holding y as a constant and vice versa.

Ex. Given $f(x,y) = x^2 + 4xy + y\sin x$

$$\frac{\partial f}{\partial x} = 2x + 4y + y\cos x, \quad \frac{\partial f}{\partial y} = 4x + \sin x$$

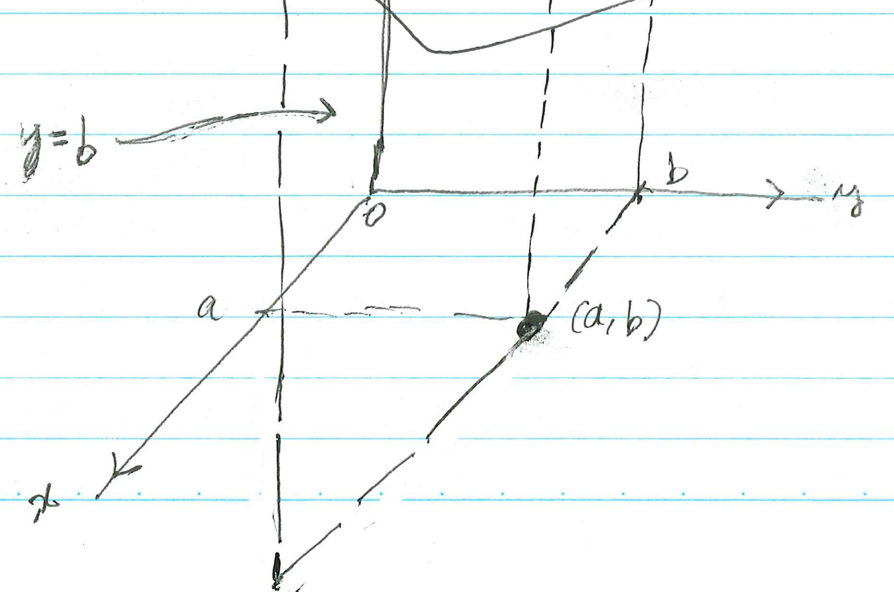
Remarks

(i) For function of three variables $f(x,y,z)$, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ are defined analogously.

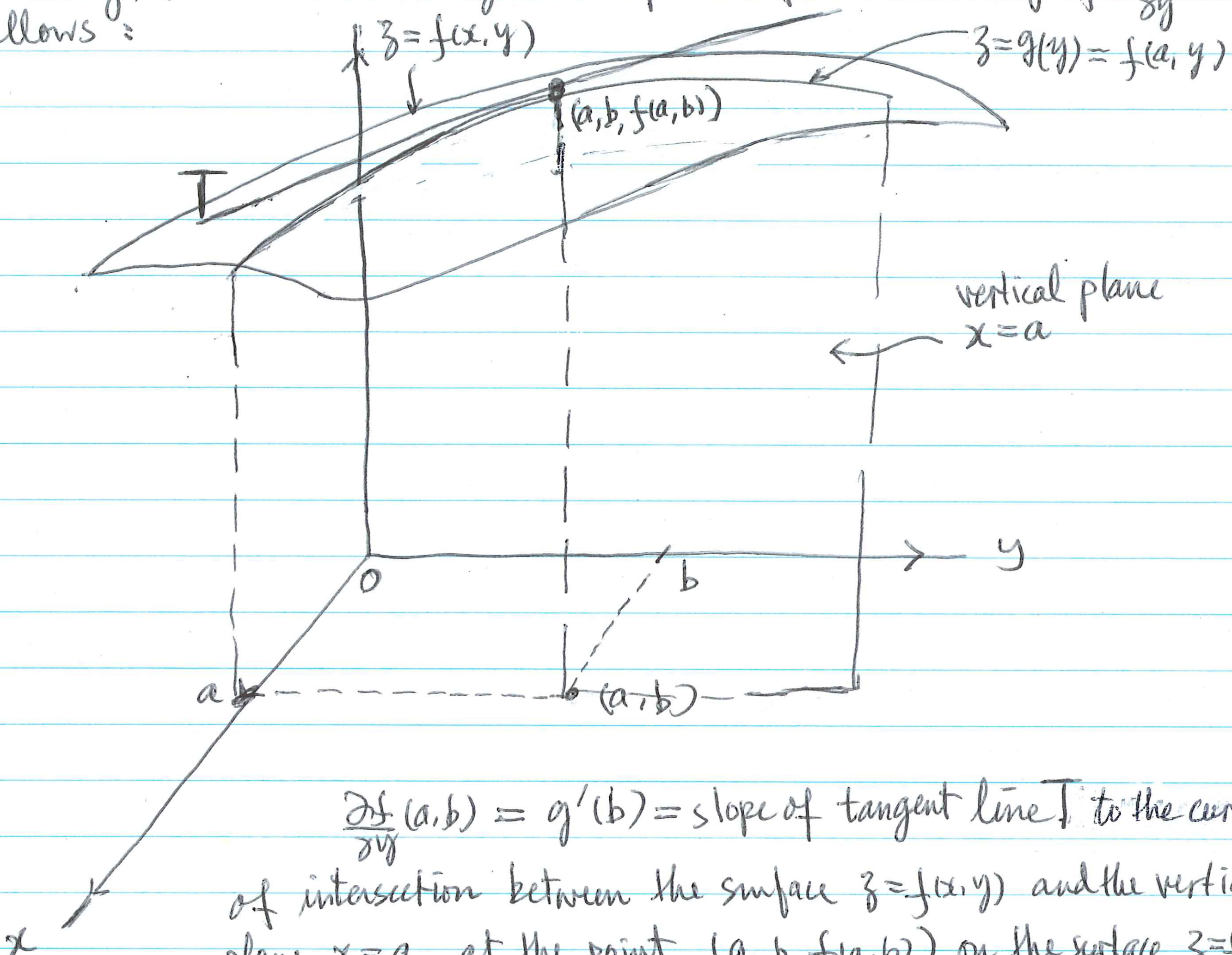
(ii) just like in the fct. of one variable case, $f'(x)$ is the rate of change of f with respect to x. $\frac{\partial f}{\partial x}$ is the rate of change of f with respect to x while the y variable is held fixed and so on.

The geometric meaning of $\frac{\partial f}{\partial x}(a,b)$ and $\frac{\partial f}{\partial y}(a,b)$ for $(a,b) \in$ domain of f

First we note that the graph of $z=f(x,y)$ is a surface in space, if we freeze the y variable to be at $y=b$, then $z=h(x)=f(x,b)$ becomes a function of x only. Its graph is the curve of intersection between the surface of $z=f(x,y)$ and the vertical plane $y=b$ (see picture). Now that $\frac{\partial f}{\partial x}(a,b) = h'(a)$ is the slope of the tangent line T to the curve of intersection $z=h(x)=f(x,b)$ at the point $(a,b,f(a,b))$ on the surface $z=f(x,y)$.



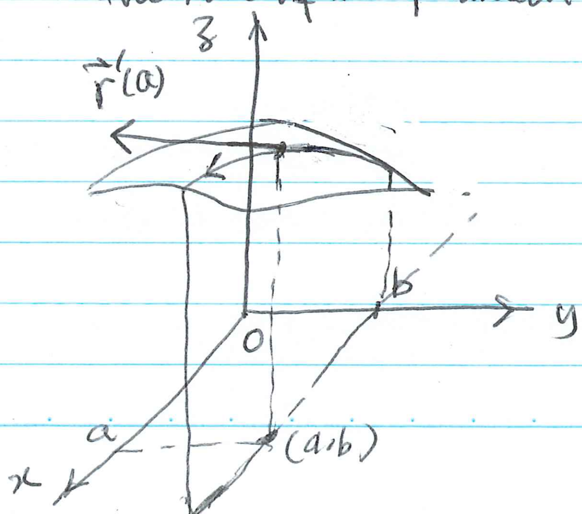
Similarly, we also have the geometric picture for the meaning of $\frac{\partial f}{\partial y}(a,b)$ as follows:



$\frac{\partial f}{\partial y}(a,b) = g'(b) =$ slope of tangent line T to the curve of intersection between the surface $z=f(x,y)$ and the vertical plane $x=a$ at the point $(a,b,f(a,b))$ on the surface $z=f(x,y)$.

Tangent Plane to the surface $z=f(x,y)$ at the point $(a,b,f(a,b))$ on the surface.

We once more define $\vec{r}(x) = \langle x, b, f(x,b)\rangle$ by freezing y at b, this is a vector-valued function which is a trajectory in space that coincides with the curve of intersection between $z=f(x,y)$ and the vertical plane $y=b$. Here x takes over the role of the parameter t. As a result,



$$\vec{r}'(a) = \langle 1, 0, \frac{\partial f}{\partial x}(a,b)\rangle$$

is tangential to the trajectory (being the velocity vector) and hence to the surface $z=f(x,y)$ at $(a,b,f(a,b))$.